B.Sc (Part II)

Subsidiary (Maths)

Leibnitz Theorem:

Statement:— If u and v are two functions of x which posses derivatives of nth order then

$$y_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_r u_{n-r} v_r + \cdots + {}^nC_n u v_n.$$

Proof:— By method of Induction

Let $y = uv$

Where u, v are functions of x.

By directly differentiating successively, we get

$$y_1 = u_1 v + u v_1$$

$$y_2 = (u_2 v + u_1 v_1) + (u v_2 + v_1 u_1)$$

$$= u_2 v + 2u_1 v_1 + u v_2$$

$$= u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 \cdot u v_2$$

$$y_3 = (u_3 v + u_2 v_1) + 2(u_1 v_2 + u_2 v_1) + (u v_3 + u_1 v_2)$$

$$= u_3 v + 3u_2 v_1 + 3u_1 v_2 + u v_3$$

$$= u_3 v + {}^3C_1 u_2 v_1 + {}^3C_2 u_1 v_2 + {}^3C_3 u v_3$$

Thus we see that this theorem is true for $n = 1, 2, 3$. According to the law of induction we assume that this theorem is true for $n = m$ and we shall prove that this will also be true for $n = m+1$, and since this is true for

Particular values of $n = 1, 2, 3, \ldots$ therefore it will be true for every value of n.

Now we assume that this theorem holds for $n = m$ i.e. we shall get the same formal expression for y_m which will be obtained by putting $n = m$ in the statement of the theorem that is

$$y_m = u_m v + {}^mc_1 u_{m-1} v_1 + {}^mc_2 u_{m-2} v_2 + \cdots$$
$$\cdots + {}^mc_{r-1} u_{m-r+1} v_{r-1} + {}^mc_r u_{m-r} v_r + \cdots$$
$$+ \cdots + {}^mc_{m-1} u_1 v_{m-1} + {}^mc_m u v_m \quad \text{---} (1)$$

Differentiating once, we get

$$y_{m+1} = (u_{m+1} v + u_m v_1) + {}^mc_1 (u_m v_1 + u_{m-1} v_2)$$
$$+ {}^mc_2 (u_{m-1} v_2 + u_{m-2} v_3) + \cdots +$$
$${}^mc_{m-1} (u_2 v_{m-1} + u_1 v_m) + {}^mc_m (u_m v_m + u v_{m+1})$$

$$= u_{m+1} v + u_m v_1 ({}^mc_0 + {}^mc_1) + u_{m-1} v_2 ({}^mc_1 + {}^mc_2) +$$
$$\cdots + u_1 v_m ({}^mc_{m-1} + {}^mc_m) + {}^mc_m u v_{m+1}$$

But we know that

$${}^mc_{r-1} + {}^mc_r = {}^{m+1}c_r$$

$\therefore$ putting $r = 1, 2, 3, \ldots$

$${}^mc_0 + {}^mc_1 = {}^{m+1}c_1, \quad {}^mc_1 + {}^mc_2 = {}^{m+1}c_2 \text{ etc}$$

Hence $y_{m+1} = u_{m+1} v + {}^{m+1}c_1 u_m v_1 + {}^{m+1}c_2 u_{m-1} v_2 +$
$\cdots + {}^{m+1}c_r u_{m-r+1} v_r + {}^{m+1}c_{m+1} u v_{m+1}.$

Thus we see that if we assume the theorem to be true for a particular value of $n=m$, then this theorem is also true for the next higher integer $n=m+1$.

But we have shown before that this theorem is true for $n=2,3$, therefore it is true for $n=4$ and since this is true for $n=4$, hence this is true for $n=5$.

Hence this theorem is true for every integer value of n.

This the theorem is proved.

Example - if $y = x^2 e^{ax}$, find y_n.

Solution:- He we take $v = x^2$ and $u = e^{ax}$

$\therefore v_1 = 2x, v_2 = 2, v_3 = 0, v_4 = 0$

$u_1 = ae^{ax}, u_2 = a^2 e^{ax}, u_3 = a^3 e^{ax}$ ——

$u_n = a^n e^{ax}$.

$\therefore$ According to Leibnitz theorem

$$y_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + {}^nC_3 u_{n-3} v_3 + \text{—}$$

$$= a^n e^{ax} x^2 + n a^{n-1} e^{ax}(2x) + \frac{n(n-1)}{1.2} a^{n-2} e^{ax}(2) + 0 + 0 + \text{—}$$

$$= a^n e^{ax} x^2 + 2n a^{n-1} x e^{ax} + n(n-1) a^{n-2} e^{ax}$$

$$= a^{n-2} e^{ax} \{ a^2 x^2 + 2nax + n(n-1) \}$$

Ans